B.1 Poisson Processes

EXPONENTIAL R. V.

$X \sim \text{Exp}(\theta = 1/\lambda)$

$f(x) = \frac{1}{\theta} e^{-x/\theta} = \lambda e^{-\lambda t}$

$\text{E}[X] = \text{ mean } = \theta,\ \lambda = 1/\theta = \text{ rate}$

$S(x) = \text{P}[X > x] = e^{-\lambda x} = e^{-x/\theta}$

Memoryless Property :

$\text{P}[X > x + a \mid X > a] = \text{P}[X > x]$

$\text{E}[X - a \mid X > a] = \text{E}[X]$

$\text{E}[X \mid X > a] = \text{E}[X] + a$

$\text{Var}[X \mid X > a] = \text{Var}[X] = \theta^2$

Sum of n Exp$(\theta) \sim$ Gammma$(\alpha = n, \theta)$

$X_i \sim \text{Exp}(\lambda_i), i = 1, \ldots, n$

$\text{P}[\min\{X_1, X_2, \ldots\} = X_i] = \frac{\lambda_i}{\lambda_1 + \lambda_2 + \ldots + \lambda_n}$

$\min\{X_1, \ldots, X_n\} \sim \text{Exp}(\theta = 1/\lambda)$

where $\lambda = \lambda_1 + \lambda_2 + \ldots + \lambda_n$

$\min\{X_1, \ldots, X_n\}$ is independent of

the ordering of the X_i.

$\max\{X_1, X_2\} = X_1 + X_2 - \min\{X_1, X_2\}$

POISSON PROCESS DEFINITION

−Counting Process

−N(0) = 0

−Independent Increments

−# events in [a,b] $\sim$ Poisson $(\lambda(b - a))$

−Stationary, if $\lambda(t) = \lambda$,

INTERARRIVAL TIMES

T_n = time between $(n-1)$st arrival and nth.

$T_n \sim$ exponential with mean $\theta = 1/\lambda$

$\text{E}[T_n] = 1/\lambda$

$\text{Var}[T_n] = 1/\lambda^2$

WAITING TIME

$S_n =$ Time until nth event.

$S_n = \sum_{i=1}^{n} T_i$

$S_n \sim \text{Gamma}\,(\alpha = n, \theta = 1/\lambda)$

$\text{P}[S_n \le t] = \text{P}[N(t) \ge n]$

$\text{E}[S_n] = n/\lambda$

$\text{Var}[S_n] = n/\lambda^2$

CLASSIFIED EVENTS

If arrivals are classified

→ Independent Poisson Processes

Ex. Next arrival type A w/ prob. p,

type B w/ prob. q

Type A arrivals

$\sim$ Poisson process with rate λp

Type B arrivals

$\sim$ Poisson process with rate λq

The two processes are independent!

P[exactly n type A before m type B]

= P[Neg. Bin.(m,p) = n]

$= \binom{n+m-1}{n} p^n q^m$

NONHOMOGENEOUS P.P.

$\lambda(t)$ varies in time

−# events in [a,b] $\sim$ Poisson $\left(\int_a^b \lambda(t)dt\right)$

COMPOUND P.P.

Each Arrival carries some random value

S = Total cost by time t

$S = \sum_{i=1}^{N} X_i;\ N \sim$ Poisson, X_i's i.i.d.

Stationary case:

$\text{E}[S] = \lambda t \text{E}[X]$,

$\text{Var}[S] = \lambda t \text{E}[X^2]$

Nonhomogeneous case:

$\text{E}[S] = \left(\int_0^t \lambda(s)ds\right) \text{E}[X]$,

$\text{Var}[S] = \left(\int_0^t \lambda(s)ds\right) \text{E}[X^2]$

SUMS of PROCESSES

$(N_1(t), \lambda_1), (N_2(t), \lambda_2)$ independent processes

$N(t) = N_1(t) + N_2(t)$ is a Poisson process

with rate $\lambda_1 + \lambda_2$

Each arrival of $N_1(t)$ has value $X_i \sim F_1(x)$

Each arrival of $N_2(t)$ has value $Y_i \sim F_2(x)$

$\sum_{i=1}^{N_1(t)} X_i + \sum_{j=1}^{N_2(t)} Y_j$ is a Compound P.P.

with rate $\lambda_1 + \lambda_2$ and arrival distribution

$F(x) = \frac{\lambda_1}{\lambda_1 + \lambda_2} F_1(x) + \frac{\lambda_2}{\lambda_1 + \lambda_2} F_2(x)$